

Graphs with multiplicative vertex-coloring 2-edge-weightings

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Abstract A k -weighting w of a graph is an assignment of an integer weight $w(e) \in \{1, \dots, k\}$ to each edge e . Such an edge weighting induces a vertex coloring c defined by $c(v) = \prod_{v \in e} w(e)$. A k -weighting of a graph G is multiplicative vertex-coloring if the induced coloring c is proper, i.e., $c(u) \neq c(v)$ for any edge $uv \in E(G)$. This paper studies the parameter $\mu(G)$, which is the minimum k for which G has a multiplicative vertex-coloring k -weighting. Chang, Lu, Wu, Yu investigated graphs with additive vertex-coloring 2-weightings (they considered sums instead of products of incident edge weights). In particular, they proved that 3-connected bipartite graphs, bipartite graphs with the minimum degree 1, and r -regular bipartite graphs with $r \geq 3$ permit an additive vertex-coloring 2-weighting. In this paper, the multiplicative version of the problem is considered. It was shown in Skowronek-Kaziów (Inf Process Lett 112:191–194, 2012) that $\mu(G) \leq 4$ for every graph G . It was also proved that every 3-colorable graph admits a multiplicative vertex-coloring 3-weighting. A natural problem to consider is whether every 2-colorable graph (i.e., a bipartite graph) has a multiplicative vertex-coloring 2-weighting. But the answer is no, since the cycle C_6 and the path P_6 do not admit a multiplicative vertex-coloring 2-weighting. The paper presents several classes of 2-colorable graphs for which $\mu(G) = 2$, including trees with at least two adjacent leaf edges, bipartite graphs with the minimum degree 3 and bipartite graphs $G = (A, B, E)$ with even $|A|$ or $|B|$.

Keywords Edge weighting · Vertex coloring · 1-2-3 Conjecture

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1 Introduction

In this paper we consider only non-trivial graphs, i.e., simple connected graphs with at least three vertices. A k -weighting of a graph is a mapping $w : E(G) \rightarrow \{1, 2, \dots, k\}$. We can define a vertex coloring $c(v)$ by $c(v) = \prod_{v \in e} w(e)$ for every $v \in V(G)$. A k -weighting of a graph G is called multiplicative vertex-coloring if for every edge uv , $c(u) \neq c(v)$.

The study of edge weightings which are additive vertex-coloring, was initiated by [Karoński et al. \(2004\)](#). A k -weighting is additive vertex-coloring if for every edge uv , the sum of weights of the edges incident to u is different than the sum of weights of the edges incident to v . They conjectured that every non-trivial graph permits an additive vertex-coloring 3-weighting (1-2-3 Conjecture), and proved this conjecture for 3-colorable graphs. The best result concerning 1-2-3 Conjecture is given by [Kalkowski et al. \(2010\)](#), who presented an additive vertex-coloring 5-weighting for every non-trivial graph.

Different versions of vertex coloring from an edge k -weighting (by considering the sums, products, sequences, sets, or multisets of incident edge weights) were investigated by many authors in [Addario-Berry et al. \(2005\)](#), [Bartnicki et al. \(2009\)](#), [Chang et al. \(2011\)](#), [Kalkowski et al. \(2010\)](#), [Karoński et al. \(2004\)](#), [Lu et al. \(2011\)](#), [Skowronek-Kaziów \(2012\)](#), [Stevens and Seamone \(2013\)](#). Some authors distinguish all the vertices in a graph by their product colors (product irregularity strength of graphs, see [Anholcer \(2009, 2014\)](#), [Darda and Hujdurovic \(2014\)](#)). The total version of this problem was also investigated, where all the edges and vertices are weighted with integers, and the color of a vertex is the sum of its weight and the weights of the incident edges (see [Przybyło and Woźniak \(2010\)](#), [Skowronek-Kaziów \(2008\)](#)). In papers [Czerwiński et al. \(2009\)](#) and [Grytczuk et al. \(2013\)](#), the vertex version of vertex-coloring was presented, in this case, only the vertices are weighted, and the color of a vertex is the sum of integer weights of its neighbors. The problem of a proper edge weighting, such that no two neighbors are adjacent to the same set of weights, was also investigated, see [Hocquard and Montassier \(2013\)](#), [Wang and Wang \(2010\)](#), [Wang \(2007\)](#).

In 2011, Chang, Lu, Wu, Yu, and Zhang [[Chang et al. \(2011\)](#), [Lu et al. \(2011\)](#)] considered the graphs with additive vertex-coloring 2-weightings. In particular, they proved that 3-connected bipartite graphs, bipartite graphs with the minimum degree 1, and r -regular bipartite graphs with $r \geq 3$ permit an additive vertex-coloring 2-weighting. In this paper, the multiplicative version of this problem is considered.

We fix the following notation. A graph C_n and P_n is a cycle and a path, respectively, with n vertices. A graph $K_{m,n}$ is a complete bipartite graph. $N[v] = N(v) \cup \{v\}$ is the closed neighborhood of a vertex v , $d(v)$ is the degree of v , and $\delta(G)$ is the minimum degree of a vertex in a graph G . The graph $G = (A, B, E)$ is a bipartite graph with edge set E and vertex bipartition (A, B) .

2 Graphs with multiplicative vertex-coloring 2-weightings

The minimum k for which G has a multiplicative vertex-coloring k -weighting is denoted by $\mu(G)$. It is easy to see that $\mu(G) \geq 2$. The following question was asked in Skowronek-Kaziów (2012):

Question (2012, Skowronek-Kaziów (2012)): Is $\mu(G) \leq 3$ for every non-trivial graph G ?

The question still remains unanswered, but the best result proved in Skowronek-Kaziów (2012) is $\mu(G) \leq 4$ for every non-trivial graph G . It was shown in Skowronek-Kaziów (2012) that every 3-colorable graph admits a multiplicative vertex-coloring 3-weighting. A natural problem to consider is whether every 2-colorable graph (i.e., a bipartite graph) has a multiplicative vertex-coloring 2-weighting. But the answer is no, since the cycle C_6 and the path P_6 do not admit a multiplicative vertex-coloring 2-weighting. Notice that a non-trivial graph G permits an additive vertex-coloring 1-weighting if and only if G has no adjacent vertices with the same degree, but there is no graph G with $\mu(G) = 1$. Of course, if G has no adjacent vertices with the same degree, then $\mu(G) = 2$.

Proposition 1 Let $G = (A, B, E)$ be a non-trivial connected bipartite graph and let $|A| \leq |B|$. Then $\mu(G) = 2$ if there exists $v_0 \in A$ such that $d(v_0) = |B|$.

Proof Let $G = (A, B, E)$ be a non-trivial bipartite graph with edge set E and vertex bipartition (A, B) , $|A| \leq |B|$. Let $v_0 \in A$ such that $d(v_0) = |B|$. We can define a 2-weighting $w : E \rightarrow \{1, 2\}$ of edges as follows: $w(v_0u) = 2$ for all $u \in B$, and $w(e) = 1$ for all remaining edges $e \in E$. Let $c(v) = \prod_{v \in e} w(e)$.

The above 2-weighting is multiplicative vertex-coloring, since $c(v_0) = 2^{|B|} \geq 4$, $c(v) = 1$ for $v \in A \setminus \{v_0\}$, and $c(u) = 2$ for all $u \in B$. \square

From Proposition 1, $\mu(K_{m,n}) = 2$.

Proposition 2 Let $G = (A, B, E)$ be a non-trivial connected bipartite graph. Then $\mu(G) = 2$ if $|A|$ or $|B|$ is even.

Proof Let $G = (A, B, E)$ be a connected bipartite graph. Let $A = \{v_1, v_2, \dots, v_{2k}\}$, $k \geq 1$. For every $i \in \{1, 2, \dots, k\}$ choose a shortest $v_i \rightarrow v_{i+k}$ path P_i . Let $\{P_1, P_2, \dots, P_k\}$ be the set of such chosen paths. Let $d_i(v)$ be the degree of a vertex v in the path P_i and let $\sigma(v) = \sum_{i=1}^k d_i(v)$ for every $v \in A \cup B$. Then $\sigma(v) \equiv 1 \pmod{2}$ for $v \in A$, and $\sigma(v) \equiv 0 \pmod{2}$ for $v \in B$. For every edge $e \in E$ let $a_i(e) = 1$ if $e \in E(P_i)$, and $a_i(e) = 0$ if $e \notin E(P_i)$. Put $\alpha(e) = \sum_{i=1}^k a_i(e)$. Now, let us define a 2-weighting $w : E \rightarrow \{1, 2\}$ as follows: $w(e) = 1$ if $\alpha(e) \equiv 0 \pmod{2}$, and $w(e) = 2$ if $\alpha(e) \equiv 1 \pmod{2}$.

Then $c(v) = 2^{2s+1}$ if $v \in A$, and $c(u) = 2^{2s}$ if $u \in B$, for some $s \geq 0$.

Hence $c(v) \neq c(u)$ for every edge $vu \in E$, $v \in A$, $u \in B$. \square

Corollary 3 *Let G be a non-trivial connected bipartite graph. Then $\mu(G) = 2$ if $|V(G)|$ is odd.*

Proposition 4 *Let G be a path or a cycle. Then $\mu(G) = 2$ if and only if G is a bipartite graph $G = (A, B, E)$ with even $|A|$ or $|B|$.*

Proof Notice that every path P_{2k} (or a cycle C_{2k}) with even $k \in \mathbb{N}$ and every path P_n with odd number n of vertices is a bipartite graph $G = (A, B, E)$ with even $|A|$ or $|B|$.

So, let G be a path P_{2k} (a cycle C_{2k}) with odd $k \in \mathbb{N}$ (which is a bipartite graph $G = (A, B, E)$ with odd $|A|$ and odd $|B|$) or a cycle C_n with odd n (which is not bipartite). Assume that there is a 2-weighting of G which is multiplicative vertex-coloring. Then, the weights should appear on consecutive edges as follows: 2,2,1,1,2,2,1,1,2,2,... . But then there are two adjacent vertices u and v with the same product colors $c(u) = c(v)$, a contradiction. \square

Proposition 5 *Let G be a non-trivial connected bipartite graph. Then $\mu(G) = 2$ if there exists a vertex with at least two neighbours of degree 1.*

Proof Let $G = (A, B, E)$ be a connected bipartite graph. By Proposition 2, we can assume that both $|A|$ and $|B|$ are odd. Assume that $d(x) = 1$ for some vertex x in A , where x is adjacent to $v \in B$ and v is adjacent to $y \in A$, $d(y) = 1$, $x \neq y$. Then $G - x = (A \setminus \{x\}, B, E \setminus \{xv\})$ is a bipartite graph with $|A \setminus \{x\}|$ even. From the proof of Proposition 2, a graph $G - x$ has a multiplicative vertex-coloring 2-weighting w such that $w(yv) = 2$. Extending this 2-weighting to G by assigning $w(xv) = 1$, we get a multiplicative vertex-coloring 2-weighting of G . \square

Proposition 6 *Let G be a non-trivial connected bipartite graph. Then $\mu(G) = 2$ if there exists a vertex x such that $d(x) \geq 3$ and a graph $G - N[x] - \{v : v \text{ is isolated in } G - N[x]\}$ is connected.*

Proof Let $G = (A, B, E)$ be a nontrivial connected bipartite graph with a vertex $x \in B$ such that $d(x) \geq 3$ and $G_1 = G - N[x] - \{v : v \text{ is isolated in } G - N[x]\}$ is connected. By Proposition 2, we may assume that both $|A|$ and $|B|$ are odd. Let $G_1 = (A_1, B_1, E_1)$, where $A_1 = A \setminus N(x)$.

Case 1 $d(x)$ is odd. In this case $|A_1|$ is even. It follows from the proof of Proposition 2 that G_1 has a 2-weighting w such that $c(v) = 2^{2s+1}$ if $v \in A_1$, and $c(v) = 2^{2s}$ if $v \in B_1$, for some $s \geq 0$. We can extend this 2-weighting to G by assigning weight 2 to each edge xu , $u \in N(x)$, and weight 1 to the remaining edges. Then it is multiplicative vertex-coloring, since $c(v) = 2^{2s}$ for $s \geq 0$ and $v \in B \setminus \{x\}$, $c(x) = 2^{d(x)} \geq 8$, $c(u) = 2$ for $u \in N(x)$.

Case 2 $d(x)$ is even. In this case $|A_1|$ is odd, and there is a vertex $u_0 \in N(x)$ adjacent to a vertex $v_0 \in B_1$. Let G' be the graph obtained from G_1 by adding the vertex u_0 and the edge u_0v_0 . It follows from the proof of Proposition 2 that G' has a 2-weighting w such that $c(v) = 2^{2s+1}$ if $v \in A_1 \cup \{u_0\}$, and $c(v) = 2^s$ if $v \in B_1$, $s \geq 0$. Notice that $w(u_0v_0) = 2$. We can extend this 2-weighting to G by assigning weight 2 to each edge xu , $u \in N(x) \setminus \{u_0\}$, and weight 1 to the remaining edges. Then it is multiplicative vertex-coloring, and $c(v) = 2^{2s}$ for $s \geq 0$ and $v \in B - \{x\}$, $c(x) = 2^{d(x)-1} \geq 8$, $c(u) = 2$ for $u \in N(x)$. \square

Corollary 7 *Let G be a non-trivial connected bipartite graph. Then $\mu(G) = 2$ if $\delta(G) \geq 3$.*

Proof Let $G = (A, B, E)$ be a connected bipartite graph with $\delta(G) \geq 3$. By Proposition 2, we may assume that both $|A|$ and $|B|$ are odd. Choose a vertex $x \in B$ such that the size of the maximum component $G_1 = (A_1, B_1, E_1)$, $A_1 \subseteq A \setminus N(x)$, of $G - N[x]$ is as large as possible. Let $G' = (A', B', E')$ be another component of $G - N[x]$, and let $x' \in B'$. Then $G_1 \cup N[x]$ is contained in a component of $G - N[x']$ which is larger than G_1 and which contradicts our choice of x . Therefore, all other components of $G - N[x]$ (except G_1) are isolated vertices in B , and the thesis follows from Proposition 6. \square

From above, all r -regular bipartite graphs, $r \geq 2$, (except cycles of length $2k$, k is odd) permit a multiplicative vertex-coloring 2-weighting.

Proposition 8 *Let G be a non-trivial connected bipartite graph, and let $x \in V(G)$ be at distance 2 from some vertex of degree 1. Then $\mu(G) = 2$ if a graph $G - x - \{v : v \text{ is isolated in } G - x\}$ is connected.*

Proof By Proposition 2, we may assume that both $|A|$ and $|B|$ are odd. Let $x \in A$ be a vertex of a path x, u_1, v_1 of length 2 where $d(v_1) = 1$. Now, let $G = (A, B, E)$ and let $G' = G - x - \{v : v \text{ is isolated in } G - x\}$ is connected.

It follows from the proof of Proposition 2 that G' has a multiplicative vertex-coloring 2-weighting w such that $w(u_1 v_1) = 2$ and $4 \leq c(u_1) = 2^{2s}$, $s \geq 1$. Now, we can extend this 2-weighting to G by changing the weight of $u_1 v_1$ from 2 to 1, assigning $w(x u_1) = 2$ and $w(e) = 1$ to all other edges e in G . In this way we get a multiplicative vertex-coloring 2-weighting of G with $c(x) = 2$, $c(u_1) \geq 4$, $c(v_1) = 1$ and $c(u) = 2^{2s}$, $s \geq 0$, for all $u \in N(x)$. \square

Every tree permits an additive vertex-coloring 2-weighting (see Lu et al. (2011)) but not always a multiplicative vertex-coloring 2-weighting (the good example is the path P_{2k} , with $\mu(P_{2k}) = 3$ for odd k).

It is a known fact that if T is a tree in which every vertex has a degree 1 or 3, then there is a vertex adjacent to two leaves.

Hence we get several sufficient conditions for a tree T with $\mu(T) = 2$:

Corollary 9 *Let T be a tree. Then $\mu(T) = 2$, if at least one of the following conditions holds:*

- T has a vertex adjacent to two leaves;*
- every vertex in T has a degree 1 or 3;*
- T has a vertex x which is at distance 2 from a leaf and $T - x - \{v : v \text{ is isolated in } T - x\}$ is connected;*
- T has an odd number of vertices.*

We can conclude the paper by the following problem:

Problem Classify all bipartite graphs (in particular trees) which permit a multiplicative vertex-coloring 2-weighting.

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